

NON-MANIFOLD MONODROMY SPACES OF BRANCHED COVERINGS BETWEEN MANIFOLDS

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ABSTRACT. By a construction of Bernstein and Edmonds every proper branched cover f between manifolds is a factor of a branched covering orbit map from a locally connected and locally compact Hausdorff space called the monodromy space of f to the target manifold. For proper branched covers between 2-manifolds the monodromy space is known to be a manifold. We show that this does not generalize to dimension 3 by constructing a self-map of the 3-sphere for which the monodromy space is not a locally contractible space.

1. INTRODUCTION

A map $f: X \rightarrow Y$ between topological spaces is a *branched covering*, if f is open, continuous and discrete map. The *branch set* $B_f \subset X$ of f is the set of points in X for which f fails to be a local homeomorphism. The map f is *proper*, if the pre-image in f of every compact set is compact.

Let $f: X \rightarrow Y$ be a proper branched covering between manifolds. Then the codimension of $B_f \subset X$ is at least two by Väisälä [14] and the restriction map

$$f' := f|_{X \setminus f^{-1}(f(B_f))}: X \setminus f^{-1}(f(B_f)) \rightarrow Y \setminus f(B_f)$$

is a covering map between open connected manifolds, see Church and Hemmingsen [6]. Thus there exists, by classical theory of covering maps, an open manifold X'_f and a commutative diagram of proper branched covering maps

$$\begin{array}{ccc} & X'_f & \\ p' \swarrow & & \searrow q' \\ X & \xrightarrow{f} & Y \end{array}$$

where $p': X'_f \rightarrow X \setminus f^{-1}(f(B_f))$ and $q': X'_f \rightarrow Y \setminus f(B_f)$ are normal covering maps and the deck-transformation group of the covering map $q': X'_f \rightarrow Y \setminus f(B_f)$ is isomorphic to the monodromy group of f' .

Further, by Bernstein and Edmonds [3], there exists a locally compact and locally connected second countable Hausdorff space $X_f \supset X'_f$ so that $X_f \setminus X'_f$ does not locally separate X_f and the maps p' and q' extend to proper

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normal branched covering maps $p: X_f \rightarrow X$ and $\bar{f} := q: X_f \rightarrow Y$ so that the diagram

$$\begin{array}{ccc} & X_f & \\ p \swarrow & & \searrow \bar{f} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, and p and \bar{f} are the Fox-completions of $p': X'_f \rightarrow X$ and $q': X'_f \rightarrow Y$, see also [1], [8] and [11]. In this paper the triple (X_f, p, \bar{f}) is called the *monodromy representation*, $\bar{f}: X_f \rightarrow Y$ the *normalization* and the space X_f the *monodromy space* of f .

The monodromy space X_f is a locally connected and locally compact Hausdorff space and, by construction, all points in the open and dense subset $X_f \setminus B_{\bar{f}} \subset X_f$ are manifold points. The natural question to ask regarding the monodromy space X_f is thus the following: *What does the monodromy space X_f look like around the branch points of \bar{f} ?*

When X and Y are 2-manifolds, Stoilows Theorem implies, that the points in $B_{\bar{f}}$ are manifold points and the monodromy space X_f is a manifold. We further know by Fox [8] that the monodromy space X_f is a locally finite simplicial complex, when $f: X \rightarrow Y$ is a simplicial branched covering between piecewise linear manifolds. It is, however, stated as a question in [8] under which assumptions the locally finite simplicial complex obtained in Fox' completion process is a manifold. We construct here an example in which the locally finite simplicial complex obtained in this way is not a manifold.

Theorem 1.1. *There exists a simplicial branched cover $f: S^3 \rightarrow S^3$ for which the monodromy space X_f is not a manifold.*

Theorem 1.1 implies that the monodromy space is not in general a manifold even for proper simplicial branched covers between piecewise linear manifolds. Our second theorem states further, that in the non-piecewise linear case the monodromy space is not in general even a locally contractible space. We construct a branched covering, which is a piecewise linear branched covering in the complement of a point, but for which the monodromy space is not a locally contractible space.

Theorem 1.2. *There exists a branched cover $f: S^3 \rightarrow S^3$ for which the monodromy space X_f is not a locally contractible space.*

We end this introduction with our results on the cohomological properties of the monodromy space. The monodromy space of a proper branched covering between manifolds is always a locally orientable space of finite cohomological dimension. However, in general the monodromy space is not a cohomology manifold in the sense of Borel [5]; there exist a piecewise linear branched covering $S^3 \rightarrow S^3$ for which the monodromy space is not a cohomology manifold. This shows, in particular, that the theory of normalization maps of proper branched covers between manifolds is not covered by Smith-theory in [5] and completes [2] for this part.

This paper is organized as follows. In Section 4, we give an example $f: S^2 \rightarrow S^2$ of an open and discrete map for which the monodromy space is not a two sphere. In Section 5 we show that the suspension $\Sigma f: S^3 \rightarrow S^3$ of

f prove Theorem 1.1 and that the monodromyspace of f is not a cohomology manifold. In Section 6 we construct an open and discrete map $g: S^3 \rightarrow S^3$. In Section 7 we show that $g: S^3 \rightarrow S^3$ proves Theorem 1.2.

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2. PRELIMINARIES

In this paper all topological spaces are locally connected and locally compact Hausdorff spaces if not stated otherwise. Further, all proper branched coverings $f: X \rightarrow Y$ between topological spaces are also branched coverings in the sense of Fox [8] and completed coverings in the sense of [1]; $Y' := Y \setminus f(B_f)$ and $X' := X \setminus f^{-1}(B_f)$ are open dense subsets so that $X \setminus X'$ does not locally separate X and $Y' \setminus Y$ does not locally separate Y . We say that the proper branched covering $f: X \rightarrow Y$ is *normal*, if $f' := f|_{X'}: X' \rightarrow Y'$ is a normal covering. By Edmonds [7] every proper normal branched covering $f: X \rightarrow Y$ is an orbit map for the action of the deck-transformation group $\text{Deck}(f)$ i.e. $X/\text{Deck}(f) \approx Y$.

We recall some elementary properties of proper branched coverings needed in the forthcoming sections. Let $f: X \rightarrow Y$ be a proper normal branched covering and $V \subset Y$ an open and connected set. Then each component of $f^{-1}(V)$ maps onto V , see [1]. Further, if the pre-image $D := f^{-1}(V)$ is connected, then $f|_D: D \rightarrow V$ is a normal branched covering and the map

$$(1) \quad \text{Deck}(f) \rightarrow \text{Deck}(f|_D), \tau \mapsto \tau|_D,$$

is an isomorphism, see [11].

Lemma 2.1. *Let $f: X \rightarrow Y$ be a branched covering between manifolds. Suppose $p: W \rightarrow X$ and $q: W \rightarrow Y$ are normal branched coverings so that $q = p \circ f$. Then $\text{Deck}(p) \subset \text{Deck}(q)$ is a normal subgroup if and only if f is a normal branched covering.*

Proof. Let $Y' := Y \setminus f(B_f)$, $X' := X \setminus f^{-1}(f(B_f))$ and $W' = W \setminus q^{-1}(f(B_f))$. Let $f' := f|_{X'}: X' \rightarrow Y'$, $p' := p|_{W'}: W' \rightarrow X'$ and $q' := q|_{W'}: W' \rightarrow Y'$ and let $w_0 \in W'$, $x_0 = p'(w_0)$ and $y_0 = q'(w_0)$. Then $\text{Deck}(p) \subset \text{Deck}(q)$ is a normal subgroup if and only if $\text{Deck}(p') \subset \text{Deck}(q')$ is a normal subgroup and the branched covering f is a normal branched covering if and only if the covering f' is normal. We have also a commutative diagram

$$\begin{array}{ccc} & W' & \\ p' \swarrow & & \searrow q' \\ X' & \xrightarrow{f'} & Y' \end{array}$$

of covering maps, where

$$q'_*(\pi_1(W', w_0)) \subset f'_*(\pi_1(X', x_0)) \subset \pi_1(Y', y_0).$$

The deck-homomorphism

$$\pi_{(q', w_0)}: \pi_1(Y', y_0) \rightarrow \text{Deck}(q')$$

now factors as

$$\begin{array}{ccc} \pi_1(Y', y_0) & \xrightarrow{\pi_{(q', w_0)}} & \text{Deck}(q') \\ & \searrow & \nearrow \\ & \pi_1(Y', y_0)/q'_*(\pi_1(W', w_0)), & \end{array}$$

for an isomorphism $\bar{\pi}_{(q', w_0)}$ and

$$\pi_{(q', w_0)}(f'_*(\pi_1(X', x_0))) = \bar{\pi}_{(q', w_0)}(f'_*(\pi_1(X', x_0))/q'_*(\pi_1(W', w_0))) = \text{Deck}(p').$$

In particular, $\text{Deck}(p') \subset \text{Deck}(q')$ is a normal subgroup if and only if

$$f'_*(\pi_1(X', x_0))/q'_*(\pi_1(W', w_0)) \subset \pi_1(Y', y_0)/q'_*(\pi_1(W', w_0))$$

is a normal subgroup. Since $q'_*(\pi_1(W', w_0)) \subset f'_*(\pi_1(X', x_0))$, this implies that $\text{Deck}(p') \subset \text{Deck}(q')$ is a normal subgroup if and only if

$$f'_*(\pi_1(X', x_0)) \subset \pi_1(Y', y_0)$$

is a normal subgroup. We conclude that f is a normal branched covering if and only if $\text{Deck}(p) \subset \text{Deck}(q)$ is a normal subgroup. \square

Let $f: X \rightarrow Y$ be a proper branched covering. We say that $D \subset X$ is a *normal neighbourhood* of x if $f^{-1}\{f(x)\} \cap D = \{x\}$ and $f|_D: D \rightarrow V$ is a proper branched covering. We note that for every $x \in X$ there exists a neighbourhood U of $f(x)$ so that for every open connected neighbourhood $V \subset U$ of $f(x)$ we have the x -component D of $f^{-1}(V)$ a normal neighbourhood of x and the pre-image $f^{-1}(E) \subset X$ is connected for every open connected subset $E \subset Y$ satisfying $Y \setminus E \subset U$. This follows from the following lemma.

Lemma 2.2. *Let $f: X \rightarrow Y$ be a proper branched covering. Then for every $y \in Y$ there exists such a neighbourhood U of y that the pre-image $f^{-1}(V) \subset X$ is connected for every open connected set $V \subset Y$ satisfying $Y \setminus V \subset U$.*

Proof. Let $y_0 \in Y \setminus \{y\}$. Since f is proper the subsets $f^{-1}\{y\}, f^{-1}\{y_0\} \subset X$ are finite. Since $W \setminus f^{-1}\{y\}$ is connected, there exists a path $\gamma: [0, 1] \rightarrow W \setminus f^{-1}\{y\}$ so that $f^{-1}\{y_0\} \subset \gamma[0, 1]$. Let $U \subset Y$ be a neighbourhood of y satisfying $U \cap f(\gamma[0, 1]) = \emptyset$.

Suppose that $V \subset Y$ is an open connected subset satisfying $Y \setminus V \subset U$. Then $f^{-1}\{y_0\} \subset \gamma[0, 1]$ is contained in a component of $f^{-1}(V)$, since $f(\gamma[0, 1]) \subset V$. Since $V \subset Y$ is connected, every component of $f^{-1}(V)$ maps onto V . Thus $f^{-1}(V) = D$. \square

We end this section with introduction the terminology and elementary results for the part of singular homology. Let X be a locally compact and locally connected second countable Hausdorff space. In this paper $H_i(X; \mathbb{Z})$ is the i :th singular homology group of X and $\tilde{H}_i(X; \mathbb{Z})$ the i :th reduced singular homology group of X with coefficients in \mathbb{Z} , see [10]. We recall that $H_i(X; \mathbb{Z}) = \tilde{H}_i(X; \mathbb{Z})$ for all $i \neq 0$ and $\tilde{H}_0(X; \mathbb{Z}) = \mathbb{Z}^{k-1}$, where k is the number of components in X . We recall that for open subsets $U, V \subset X$ with $X = U \cup V$ and $X = U \cap V$ connected the reduced Mayer-Vietoris sequence is a long exact sequence of homomorphisms that terminates as follows:

$$\rightarrow H_1(X; \mathbb{Z}) \rightarrow \tilde{H}_0(U \cap V; \mathbb{Z}) \rightarrow \tilde{H}_0(U; \mathbb{Z}) \oplus \tilde{H}_0(V; \mathbb{Z}) \rightarrow \tilde{H}_0(X; \mathbb{Z}).$$

3. LOCAL ORIENTABILITY AND COHOMOLOGICAL DIMENSION

In this section we show that the monodromy space of a proper branched covering between manifolds is a locally orientable space of finite cohomological dimension. We also introduce Alexander-Spanier cohomology following the terminology of Borel [5] and Massey [10] and define a cohomology manifold in the sense of Borel [5].

Let X be a locally compact and locally connected second countable Hausdorff space. In this paper $H_c^i(X; \mathbb{Z})$ is the i :th Alexander-Spanier cohomology group of X with coefficients in \mathbb{Z} and compact supports. Let $A \subset X$ be a closed subset and $U = X \setminus A$. The standard push-forward homomorphism $H_c^i(U; \mathbb{Z}) \rightarrow H_c^i(X; \mathbb{Z})$ is denoted τ_{XU}^i , the standard restriction homomorphism $H_c^i(X; \mathbb{Z}) \rightarrow H_c^i(U; \mathbb{Z})$ is denoted ι_{UX}^i and the standard boundary homomorphism $H_c^i(A; \mathbb{Z}) \rightarrow H_c^{i+1}(X \setminus A; \mathbb{Z})$ is denoted $\partial_{(X/A)A}^i$ for all $i \in \mathbb{N}$.

We recall that the exact sequence of the pair (X, A) is a long exact sequence

$$\rightarrow H_c^i(X \setminus A; \mathbb{Z}) \rightarrow H_c^i(X; \mathbb{Z}) \rightarrow H_c^i(A; \mathbb{Z}) \rightarrow H_c^{i+1}(X \setminus A; \mathbb{Z}) \rightarrow$$

where all the homomorphisms are the standard ones. We also recall that $\tau_{VX}^i = \tau_{XU}^i \circ \tau_{UV}^i$ for all open subsets $V \subset U$ and $i \in \mathbb{N}$.

The *cohomological dimension* of a locally compact and locally connected Hausdorff space X is $\leq n$, if $H_c^{n+1}(U; \mathbb{Z}) = 0$ for all open subsets $U \subset X$.

Theorem 3.1. *Let $f: X \rightarrow Y$ be a proper branched covering between n -manifolds. Then the monodromy space X_f of f has dimension $\leq n$.*

Proof. Let $B_{\bar{f}} \subset X_f$ be the branch set of the normalization map $\bar{f}: X_f \rightarrow Y$ of f . Let $U \subset X_f$ be a connected open subset and $B_{\bar{f}|U}$ the branch set of $\bar{f}|U$. The cohomological dimension of $B_{\bar{f}|U}$ is at most $n-2$ by [5], since $B_{\bar{f}|U}$ does not locally separate U . Thus $H_c^i(B_{\bar{f}|U}; \mathbb{Z}) = 0$ for $i > n-2$ and the part

$$\rightarrow H_c^{i-1}(B_{\bar{f}|U}; \mathbb{Z}) \rightarrow H_c^i(U \setminus B_{\bar{f}|U}; \mathbb{Z}) \rightarrow H_c^i(U; \mathbb{Z}) \rightarrow H_c^i(B_{\bar{f}|U}; \mathbb{Z}) \rightarrow$$

of the long exact sequence of the pair $(U, B_{\bar{f}|U})$ gives us an isomorphism $H_c^i(U \setminus B_{\bar{f}|U}; \mathbb{Z}) \rightarrow H_c^i(U; \mathbb{Z})$ for $i \geq n$. Since $U \setminus B_{\bar{f}|U}$ is a connected n -manifold, $H_c^{n+1}(U \setminus B_{\bar{f}|U}; \mathbb{Z}) = 0$. Thus $H_c^{n+1}(U; \mathbb{Z}) \cong H_c^{n+1}(U \setminus B_{\bar{f}|U}; \mathbb{Z}) = 0$. We conclude that X_f has dimension $\leq n$. \square

The i :th *local Betti-number* $\rho^i(x)$ around x is k , if given a neighbourhood U of x , there exists open neighbourhoods $W \subset V \subset U$ with $\bar{W} \subset V$ and $\bar{V} \subset U$ so that $\text{Im}(\tau_{VW'}^i) = \text{Im}(\tau_{VW'}^i)$ and has rank k . The space X is called a *Wilder manifold*, if X is finite dimensional and for all $x \in X$ the local Betti-numbers satisfy $\rho^i(x) = 0$ for all $i < n$ and $\rho^n(x) = 1$.

A locally compact and locally connected Hausdorff space X with cohomological dimension $\leq n$ is *orientable*, if there exists for every $x \in X$ a neighbourhood basis \mathcal{U} of x so that $\text{Im}(\tau_{XU}^n) = \mathbb{Z}$ for all $U \in \mathcal{U}$, and locally orientable if every point in X has an orientable neighbourhood.

Theorem 3.2. *Let $\bar{f}: X_f \rightarrow Y$ be a normalization map of a branched covering $f: X \rightarrow Y$ so that Y is orientable. Then X_f is orientable.*

Proof. The set $Y \setminus \bar{f}(B_{\bar{f}})$ is an open connected subset of the orientable manifold Y . Thus $X_f \setminus B_{\bar{f}}$ is an orientable manifold as a cover of the orientable manifold $Y \setminus \bar{f}(B_{\bar{f}})$. Thus $H_c^n(X'_f; \mathbb{Z}) = \mathbb{Z}$.

We show that $H_c^n(X_f; \mathbb{Z}) = \mathbb{Z}$ and that the push-forward $\tau_{X_f W}$ is an isomorphism for every $x \in X_f$ and normal neighbourhood W of x . The cohomological dimension of $B_{\bar{f}}$ is ≤ 2 . Thus, by the long exact sequences of the pairs $(X_f, B_{\bar{f}})$ and $(W, B_{\bar{f}} \cap W)$, the push-forward homomorphisms $\tau_{X_f(X_f \setminus B_{\bar{f}})}^n$ and $\tau_{W(W \setminus B_{\bar{f}})}^n$ are isomorphisms. Since $W \setminus B_{\bar{f}} \subset X_f \setminus B_{\bar{f}}$ is a connected open subset and $X_f \setminus B_{\bar{f}}$ is a connected orientable n -manifold, the push-forward homomorphism $\tau_{(X_f \setminus B_{\bar{f}})((W \setminus B_{\bar{f}}))}^n$ is an isomorphism. Since $\tau_{X_f W}^n \circ \tau_{W(W \setminus B_{\bar{f}})}^n = \tau_{X_f(X_f \setminus B_{\bar{f}})}^n \circ \tau_{(X_f \setminus B_{\bar{f}})(W \setminus B_{\bar{f}})}^n$, we conclude that $\tau_{X_f W}^n$ is an isomorphism and $\text{Im}(\tau_{X_f W}^n) = \text{Im}(\tau_{X_f X'_f}^n) \cong \mathbb{Z}$. \square

We note that a similar argument shows that a monodromy space of a proper branched covering between manifolds is always locally orientable. A *cohomology manifold* in the sense of Borel [5] is a locally orientable Wilder manifold.

4. THE MONODROMY SPACE OF BRANCHED COVERS BETWEEN SURFACES

A *surface* is a closed orientable 2-manifold. The monodromy space related to a branched covering between surfaces is always a surface as mentioned in [8]. We first present the proof of this fact in the case we use it for completion of presentation and then we show that there exists a branched cover $f: S^2 \rightarrow S^2$ so that $X_f \neq S^2$ towards proving Theorems 1.1 and 5.1.

Lemma 4.1. *Let F be an orientable surface and $f: F \rightarrow S^2$ be a proper branched cover and $\bar{f}: X_f \rightarrow S^2$ the normalization of f . Then X_f is an orientable surface.*

Proof. Since S^2 is orientable, the space X_f is orientable by Theorem 3.2. Since the domain F of f is compact, the normalization \bar{f} has finite multiplicity and the space X_f is compact. Let $x \in X_f$. By Stoilow's theorem, see [15], $f(B_f) = \bar{f}(B_{\bar{f}})$ is a discrete set of points. Thus there exists a normal neighbourhood $V \subset X_f$ of x so that $\bar{f}(V) \cap f(B_f) \subset \{f(x)\}$ and $\bar{f}(V) \approx \mathbb{R}^2$. Now $\bar{f}|_{V \setminus \{x\}}: V \setminus \{x\} \rightarrow \bar{f}(V \setminus \{f(x)\})$ is a cyclic covering of finite multiplicity, since $\bar{f}(V \setminus \{f(x)\})$ is homeomorphic to the complement of a point in \mathbb{R}^2 . We conclude from this that x is a manifold point of X_f . Thus X_f is a 2-manifold and a surface. \square

We record as a theorem the following result in the spirit of Fox [8, p.255].

Theorem 4.2. *Let F be an orientable surface and $f: F \rightarrow S^2$ a proper branched cover and $\bar{f}: X_f \rightarrow S^2$ the normalization of f . Assume $|fB_f| > 3$. Then $X_f \neq S^2$.*

Proof. The space X_f is S^2 if and only if the Euler characteristics $\chi(X_f)$ is 2. By Riemann Hurwitz formula

$$\chi(X_f) = (\deg \bar{f})\chi(S^2) - \sum_{x \in X_f} (i(x, \bar{f}) - 1),$$

where $i(x, \bar{f})$ is the local index of \bar{f} at x . Since \bar{f} is a normal branched cover, $i(x', \bar{f}) = i(x, \bar{f})$ for $x, x' \in X_f$ with $\bar{f}(x) = \bar{f}(x')$. We define for all $y \in S^2$,

$$n(y) := i(x, \bar{f}), x \in \bar{f}^{-1}\{y\}.$$

Then for all $y \in S^2$

$$\deg \bar{f} = \sum_{x \in \bar{f}^{-1}\{y\}} i(x, \bar{f}) = n(y) |\bar{f}^{-1}\{y\}|$$

and thus for all $y \in S^2$

$$|\bar{f}^{-1}\{y\}| = \frac{\deg \bar{f}}{n(y)}.$$

Hence,

$$\chi(X_f) = (\deg \bar{f}) \left(\chi(S^2) - \sum_{y \in \bar{f}B_{\bar{f}}} \frac{n(y) - 1}{n(y)} \right),$$

where $\chi(S^2) = 2$ and $(\deg \bar{f}) := N \in \mathbb{N}$. Since $n(y) \geq 2$ for all $y \in \bar{f}B_{\bar{f}} = fB_f$ and $\frac{k-1}{k} \rightarrow 1$ as $k \rightarrow \infty$, we get the estimate

$$\chi(X_f) \leq N \left(2 - |fB_f| \frac{1}{2} \right).$$

Thus $\chi(X_f) \leq 0 < 2$, since $|fB_f| \geq 4$ by assumption. Thus $X_f \neq S^2$. \square

We end this section with two independent easy corollaries.

Corollary 4.3. *Let F be an orientable surface and $f: F \rightarrow S^2$ be a proper branched cover so that $|fB_f| > 3$. Then f is not a normal covering.*

Corollary 4.4. *Let F be an orientable surface and $f: F \rightarrow S^2$ be a proper branched cover so that $|fB_f| < 3$. Then f is a normal covering.*

Proof. Since the first fundamental group of $S^2 \setminus fB_f$ is cyclic, the monodromy group of $f|_{F \setminus f^{-1}(fB_f)}: F \setminus f^{-1}(fB_f) \rightarrow S^2 \setminus fB_f$ is cyclic. Thus every subgroup of the deck-transformation group of the normalization map $\bar{f}: X_f \rightarrow X$ is a normal subgroup. Thus $f = \bar{f}$ and in particular, f is a normal branched covering. \square

5. THE SUSPENSION OF A BRANCHED COVER BETWEEN ORIENTABLE SURFACES

In this section we prove Theorem 1.1 in the introduction and the following theorem.

Theorem 5.1. *There exists a simplicial branched cover $f: S^3 \rightarrow S^3$ for which the monodromy space X_f is not a cohomology manifold.*

More precisely, we show that there exists a branched cover $S^2 \rightarrow S^2$ for which the monodromy space is not a manifold or a cohomology manifold for the suspension map $\Sigma S^2 \rightarrow \Sigma S^2$.

Let F be an orientable surface. Let \sim be the equivalence relation in $F \times [-1, 1]$ defined by the relation $(x, t) \sim (x', t)$ for $x, x' \in F$ and $t \in \{-1, 1\}$. Then the quotient space $\Sigma F := F \times [-1, 1] / \sim$ is the *suspension space* of F

and the subset $CF := \overline{\{(x, t) : x \in F, t \in [0, 1]\}} \subset \Sigma F$ is the *cone* of F . We note that $\Sigma S^2 \approx S^3$. Let $f: F_1 \rightarrow F_2$ be a piecewise linear branched cover between surfaces. Then $\Sigma f: \Sigma F_1 \rightarrow \Sigma F_2, (x, t) \mapsto (f(x), t)$, is a piecewise linear branched cover and called the *suspension map* of f . We note that the suspension space ΣF is a polyhedron and locally contractible for all surfaces F .

We begin this section with a lemma showing that the normalization of a suspension map of a branched cover between surfaces is completely determined by the normalization of the original map.

Lemma 5.2. *Let F be an orientable surface and $f: F \rightarrow S^2$ a branched cover and $\bar{f}: X_f \rightarrow S^2$ the normalization of f . Then $\Sigma \bar{f}: \Sigma X_f \rightarrow \Sigma S^2$ is the normalization of $\Sigma f: \Sigma F \rightarrow \Sigma S^2$.*

Proof. Let $p: X_f \rightarrow F$ be a normal branched covering so that $\bar{f} = f \circ p$. Then $\Sigma \bar{f}$ is a normal branched cover so that $\Sigma \bar{f} = \Sigma p \circ \Sigma f$ and $\varphi: \text{Deck}(\Sigma \bar{f}) \rightarrow \text{Deck}(\bar{f}), \tau \mapsto \tau|_{X_f}$ is an isomorphism. We need to show that, if $G \subset \text{Deck}(\Sigma \bar{f})$ is a subgroup so that $f \circ (\Sigma p/G)$ is normal. Then G is trivial.

Suppose $G \subset \text{Deck}(\Sigma \bar{f})$ is a group so that $f \circ \Sigma/G$ is normal. Then $f \circ (p/G')$ is normal for the quotient map $p/G': X_f/G' \rightarrow S^2$, where $G' = \varphi(G)$. Since \bar{f} is the normalization of f , the group G' is trivial. Thus $G = \varphi^{-1}(G')$ is trivial, since φ^{-1} is an isomorphism. \square

We then characterize the surfaces for which the suspension space is a manifold or a cohomology manifold in the sense of Borel.

Lemma 5.3. *Let F be an orientable surface. Then ΣF is a manifold if and only if $F = S^2$.*

Proof. Suppose $F = S^2$. Then $\Sigma F \approx S^3$. Suppose then that $F \neq S^2$. Then there exists a (cone) point $x \in \Sigma F$ and a contractible neighbourhood $V \subset \Sigma F$ of x so that $F \subset V$ and $V \setminus \{x\}$ contracts to F . Now $\pi_1(V \setminus \{x\}, x_0) \cong \pi_1(F, x_0) \neq 0$ for $x_0 \in F$. Suppose that ΣF is a 3-manifold. Then $\pi_1(V \setminus \{x\}, x_0) = \pi_1(V, x_0) = 0$, which is a contradiction. Thus ΣF is not a manifold. \square

Lemma 5.4. *Let F be an orientable surface. Then ΣF is not a Wilder manifold if $F \neq S^2$.*

Proof. We show that then the second local Betti-number is non-trivial around a point in ΣF . Let $CF \subset \Sigma F$ be the cone of F . Let $\pi: F \times [0, 1] \rightarrow \Sigma F, (x, t) \mapsto (x, t)$, be the quotient map to the suspension space and $\bar{x} = \pi(F \times \{1\})$. We first note that $H_c^1(CF, \mathbb{Z}) = H_c^2(CF, \mathbb{Z}) = 0$, since CF contracts properly to a point. Further, by Poincaré duality $H_c^1(F; \mathbb{Z}) = \mathbb{Z}^{2g}$, where g is the genus of F . In the exact sequence of the pair (CF, F) we have the short exact sequence

$$\rightarrow 0 \rightarrow H_c^2(CF \setminus F; \mathbb{Z}) \rightarrow H_c^1(F; \mathbb{Z}) \rightarrow 0.$$

Thus $H_c^2(CF \setminus F; \mathbb{Z}) \cong H_c^1(F; \mathbb{Z})$ and $H_c^2(CF \setminus F; \mathbb{Z}) = 0$ if and only if $g = 0$ for the genus g of F . Thus $H_c^2(CF \setminus F; \mathbb{Z}) = 0$ if and only if $F = S^2$.

We then show that the rank of $H_c^2(CF \setminus F; \mathbb{Z})$ is the local Betti-number $\rho^2(\bar{x})$ around \bar{x} . For this it is sufficient to show that given any neighbourhood

$U \subset CF$ of \bar{x} , there exists open neighbourhoods $W \subset V \subset U$ of \bar{x} with $\bar{W} \subset V, \bar{V} \subset U$, so that for any open neighbourhood $W' \subset W$ of \bar{x} , $\text{Im}(\tau_{VW}) = \text{Im}(\tau_{VW'}) \cong H_c^2(CF \setminus F; \mathbb{Z})$.

Denote $\Omega_t = \varphi(F \times [0, t))$ for all $t \in (0, 1)$. We note that then $\tau_{\Omega_s \Omega_t}$ is an isomorphism for all $t, s \in \mathbb{R}, t < s$, since $\iota: \Omega_t \rightarrow \Omega_s$ is properly homotopic to the identity. Let $U \subset CF$ be any neighbourhood of \bar{x} . We set $V = \Omega_t$ for such $t \in (0, 1)$ that $\Omega_t \subset U$ and $W = \Omega_{t/2}$. Then for any neighbourhood $W' \subset \Omega_t$ of \bar{x} there exists $t' \in (0, t/2)$ so that $\Omega_{t'} \subset W'$. Now $\tau_{\Omega_t W'}$ is surjective, since $\tau_{\Omega_t \Omega_{t'}} = \tau_{\Omega_t W'} \circ \tau_{W' \Omega_{t'}}$ is an isomorphism. Thus

$$\text{Im}(\tau_{\Omega_t \Omega_{t/2}}) = \text{Im}(\tau_{\Omega_t \Omega_{t'}}) = H_c^2(\Omega_t; \mathbb{Z}) \cong H_c^2(CF \setminus F; \mathbb{Z}),$$

since $\tau_{\Omega_t \Omega_{t/2}}$ and $\tau_{\Omega_t(CF/F)}$ are isomorphisms. Thus $\rho^2(\bar{x}) \neq 0$, since $F \neq S^2$. \square

Corollary 5.5. *Let $f: S^2 \rightarrow S^2$ be a branched cover with $|fB_f| > 3$, $\Sigma f: S^3 \rightarrow S^3$ the suspension map of f and $\bar{\Sigma}f: X_{\Sigma f} \rightarrow S^3$ the normalization of Σf . Then $X_{\Sigma f}$ is not a manifold and not a Wilder manifold.*

Proof. By Lemmas 4.1 and 5.2 we know that X_f is a surface and $X_{\Sigma f} = \Sigma X_f$. Further, by Lemma 4.2 we know that $X_f \neq S^2$, since $|fB_f| > 3$. Thus, by Lemma 5.3, X_f is not a manifold. Further, by Lemma 5.4, $X_{\Sigma f}$ is not a Wilder manifold. Thus X_f is not a cohomology manifold in the sense of Borel. \square

Proof of Theorems 1.1 and 5.1. By Corollary 5.5 it is sufficient to show that there exists a branched cover $f: S^2 \rightarrow S^2$ so that $|f(B_f)| = 4$. Such a branched cover we may easily construct as $f = f_1 \circ f_2$, where $f_i: S^2 \rightarrow S^2$ is a winding map with branch points x_1^i and x_2^i for $i \in \{1, 2\}$ satisfying $x_1^2, x_2^2 \notin \{f_1(x_1^1), f_1(x_2^1)\}$ and $f_2(f_1(x_1^1)) \neq f_2(f_1(x_2^1))$. \square

6. AN EXAMPLE OF A NON-LOCALLY CONTRACTIBLE MONODROMY SPACE

In this section we introduce an example of a branched cover $S^3 \rightarrow S^3$ for which the related monodromy space is not a locally contractible space. The construction of the example is inspired by Heinones and Rickmans construction in [9] of a branched covering $S^3 \rightarrow S^3$ containing a wild Cantor set in the branch set. We need the following result originally due to Bernstein and Edmonds [4] in the extent we use it.

Theorem 6.1 ([12], Theorem 3.1). *Let W be a connected, compact, oriented piecewise linear 3-manifold whose boundary consists of $p \geq 2$ components M_0, \dots, M_{p-1} with the induced orientation. Let $W' = N \setminus \text{int}B_j$ be an oriented piecewise linear 3-sphere N in \mathbb{R}^4 with p disjoint, closed, polyhedral 3-balls removed, and have the induced orientation on the boundary. Suppose that $n \geq 3$ and $\varphi_j: M_j \rightarrow \partial B_j$ is a sense preserving piecewise linear branched cover of degree n , for each $j = 0, 1, \dots, p-1$. Then there exists a sense preserving piecewise linear branched cover $\varphi: W \rightarrow W'$ of degree n that extends $\varphi_j: s$.*

Let $x \in S^3$ be a point in the domain and $y \in S^3$ a point in the target. Let $X \subset S^3$ be a closed piecewise linear ball with center x and let $Y \subset S^3$

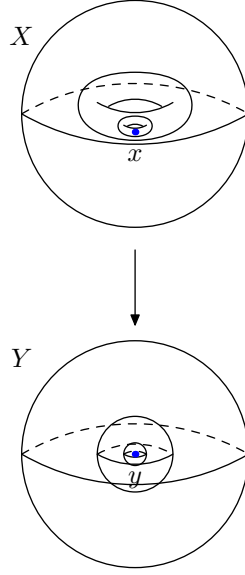


FIGURE 1.

be a closed piecewise linear ball with center y . Let $T_0 \subset \text{int}X$ be a solid piecewise linear torus so that $x \in \text{int}T_0$. Now let $\mathcal{T} = (T_n)_{n \in \mathbb{N}}$ be a sequence of solid piecewise linear tori in T_0 so that $T_{k+1} \subset \text{int}T_k$ for all $k \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} T_n = \{x\}$. Let further $B_0 \subset \text{int}Y$ be a closed piecewise linear ball with center y and let $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a sequence of closed piecewise linear balls with center y so that $B_{k+1} \subset \text{int}B_k$ for all $k \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} B_n = \{y\}$. See illustration in Figure 1.

We denote $\partial X = \partial T_{-1}$ and $\partial Y = \partial B_{-1}$ and choose an orientation to all boundary surfaces from an outward normal. Let $f_n: \partial T_n \rightarrow \partial B_n, n \in \{-1\} \cup \mathbb{N}$, be a collection of sense preserving piecewise linear branched coverings so that

- (i) the degree of all the maps in the collection are the same and greater than 2,
- (ii) f_{-1} has an extension to a branched covering $g: S^3 \setminus \text{int}X \rightarrow S^3 \setminus \text{int}Y$,
- (iii) the maps f_n are for all even $n \in \mathbb{N}$ normal branched covers with no points of local degree three and
- (iv) the branched covers f_n have for all uneven $n \in \mathbb{N}$ a point of local degree three.

We note that for an example of such a collection of maps of degree 18, we may let f_{-1} be a 18-to-1 winding map, f_n be for even $n \in \mathbb{N}$ as illustrated in Figure 2 and f_n be for all uneven $n \in \mathbb{N}$ as illustrated in Figure 3.

Let then $n \in \mathbb{N}$ and let $F_n \subset X$ be the compact piecewise linear manifold with boundary $\partial T_{n-1} \cup \partial T_n$ that is the closure of a component of $X \setminus (\bigcup_{k=-1}^{\infty} \partial T_k)$. Let further, $G_n \subset Y$ be the compact piecewise linear manifold with boundary $\partial B_{n-1} \cup \partial B_n$ that is the closure of a component of $Y \setminus (\bigcup_{k=-1}^{\infty} \partial B_k)$. Then $F_n \subset X$ is a compact piecewise linear manifold with two boundary components and $G_n \subset Y$ is the complement of the interior of two distinct piecewise linear balls in S^3 . Further, $f_{n-1}: \partial T_{n-1} \rightarrow \partial B_{n-1}$

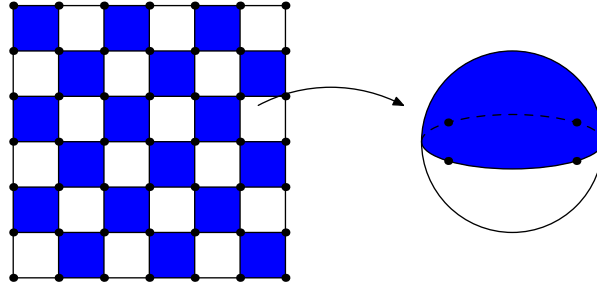


FIGURE 2.

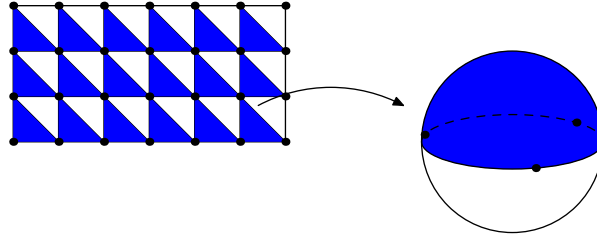


FIGURE 3.

and $f_n: \partial T_n \rightarrow \partial B_n$ are sense preserving piecewise linear branched covers between the boundary components of F_n and G_n . Since the degree of f_n is the same as the degree of f_{n-1} and the degree is greater than 2, there exists by 6.1 a piecewise linear branched cover $g_n: F_n \rightarrow G_n$ so that $g_n|_{\partial T_{n-1}} = f_{n-1}$ and $g_n|_{\partial T_n} = f_n$.

Now $X = \bigcup_{k=0}^{\infty} F_n$ and $Y = \bigcup_{k=0}^{\infty} G_n$ and $g: S^3 \setminus \text{int} X \rightarrow S^3 \setminus \text{int} Y$ satisfies $g|_{\partial X} = g_0|_{\partial X}$. Hence we may define a branched covering $f: S^3 \rightarrow S^3$ by setting $f(x) = g_n(x)$ for $x \in G_n, n \in \mathbb{N}$, and $f(x) = g(x)$ otherwise.

However, we want the map $f: S^3 \rightarrow S^3$ to satisfy one more technical condition, namely the existence of collections of properly disjoint open sets $(M_k)_{k \in \mathbb{N}}$ of X and $(N_k)_{k \in \mathbb{N}}$ of Y so that $M_k \subset X$ is a piecewise linear regular neighbourhood of ∂T_k and $N_k \subset Y$ is a piecewise linear regular neighbourhood of B_k and $M_k = f^{-1}N_k$, and $f|_{M_k}: M_k \rightarrow N_k$ has a product structure of f_k and the identity map for all $k \in \mathbb{N}$. We may require this to hold for the $f: S^3 \rightarrow S^3$ defined, since in other case we may by cutting S^3 along the boundary surfaces of ∂T_k and ∂B_k and adding regular neighbourhoods M_k of ∂T_k and N_k of ∂B_k in between for all $k \in \mathbb{N}$ arrange this to hold without loss of conditions (i)–(iv), see [13].

In the last section of this paper we prove the following theorem.

Theorem 6.2. *Let $f: S^3 \rightarrow S^3$ and $y \in S^3$ be as above and $\bar{f}: X_f \rightarrow Y$ the normalization of f . Then $H_1(W; \mathbb{Z}) \neq 0$ for all open sets $W \subset X_f$ satisfying*

$$\bar{f}^{-1}\{y\} \cap W \neq \emptyset.$$

Theorem 1.1 in the introduction then follows from Theorem 6.2 by the following easy corollary.

Corollary 6.3. *Let $f: S^3 \rightarrow S^3$ and $y \in S^3$ be as above. Then the monodromy space X_f of f is not locally contractible.*

Proof. Let $x \in \bar{f}^{-1}\{y\}$ and W a neighbourhood of x . Then $H_1(W; \mathbb{Z}) \neq 0$ and W has non-trivial fundamental group by Hurewicz Theorem, see [10]. Thus W is not contractible. Thus the monodromy space X_f of f is not a locally contractible space. \square

7. DESTRUCTIVE POINTS

In this section we define destructive points and prove Theorem 6.2.

Let X be a locally connected Hausdorff space. We call an open and connected subset $V \subset X$ a *domain*. Let $V \subset X$ be a domain. A pair $\{A, B\}$ is called a *domain covering* of V , if $A, B \subset X$ are domains and $V = A \cup B$. We say that a domain covering $\{A, B\}$ of V is *strong*, if $A \cap B$ is connected. Let $x \in V$ and let $U \subset V$ be a neighbourhood of x . Then we say that $\{A, B\}$ is *U -small* at x , if $x \in A \subset U$ or $x \in B \subset U$.

Let then $f: X \rightarrow Y$ be a branched covering between manifolds, $y \in Y$ and $V_0 \subset Y$ a domain containing y . Then V_0 is a *destructive neighbourhood* of y with respect to f , if $f|f^{-1}(V_0)$ is not a normal covering to its image, but there exists for every neighbourhood $U \subset V_0$ of y a U -small strong domain covering $\{A, B\}$ of V_0 at y so that $\{f^{-1}(A), f^{-1}(B)\}$ is a strong domain cover of $f^{-1}(V_0)$ and $f|(f^{-1}(A) \cap f^{-1}(B))$ is a normal covering to its image.

We say that y is a *destructive point* of f , if y has a neighbourhood basis consisting of neighbourhoods that are destructive with respect to f .

Theorem 7.1. *The map $S^3 \rightarrow S^3$ of the example in section 6 has a destructive point.*

Proof. We show that $y \in \bigcap_{n=1}^{\infty} B_n$ is a destructive point of f . We first show that $V_0 = \text{int} B_0$ is a destructive neighbourhood of y .

We begin this by showing that $g := f|f^{-1}(V_0): f^{-1}(V_0) \rightarrow V_0$ is not a normal branched cover. Towards contradiction suppose that g is a normal branched cover. Then $\text{Deck}(g) \cong \text{Deck}(g|M_1)$ and $\text{Deck}(g) \cong \text{Deck}(g|M_2)$, since $M_1 = f^{-1}(N_1)$ and $M_2 = f^{-1}(N_2)$ are connected. On the other hand (iii) and (iv) imply that $\text{Deck}(g|M_1) \not\cong \text{Deck}(g|M_2)$ and we have a contradiction.

Let then $V_1 \subset V_0$ be any open connected neighbourhood of y . Then there exists such $k \in \mathbb{N}$, that $B_{2k} \cup N_{2k} \subset V_1$. Let $B := B_{2k} \cup N_{2k}$ and $A = (V_0 \setminus B_{2k}) \cup N_{2k}$. Then $\{A, B\}$ is a strong V_1 -small domain cover of V_0 at y and $A \cap B = N_{2k}$. In particular, $\{f^{-1}(A), f^{-1}(B)\}$ is a strong domain cover of $f^{-1}(V_0)$. Further,

$$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(N_{2k}) = M_{2k}$$

and $f|(f^{-1}(A) \cap f^{-1}(B)) = f|M_{2k}: M_{2k} \rightarrow N_{2k}$ is a normal branched covering by (iii). Thus V_0 is a destructive neighbourhood of y . The same argument shows that $V_k := \text{int} B_k$ is a destructive neighbourhood of y for all $k \in \mathbb{N}$. Thus y has a neighbourhood basis consisting of neighbourhoods that are destructive with respect to f . \square

Theorem 7.1 implies that Theorems 6.2 and 1.1 follow from the following result.

Theorem 7.2. *Let $f: X \rightarrow Y$ be a proper branched covering between manifolds and let*

$$\begin{array}{ccc} & X_f & \\ p \swarrow & & \searrow \bar{f} \\ X & \xrightarrow{f} & Y \end{array}$$

be a commutative diagram of branched coverings so that X_f is a connected, locally connected Hausdorff space and $p: X_f \rightarrow X$ and $q: X_f \rightarrow Y$ are proper normal branched coverings. Suppose there exists a destructive point $y \in Y$ of f . Then $H_1(W; \mathbb{Z}) \neq 0$ for all open sets $W \subset X_f$ satisfying

$$\bar{f}^{-1}\{y\} \cap W \neq \emptyset.$$

We begin the proof of Theorem 7.2 with two lemmas. The following observation is well known for experts.

Lemma 7.3. *Let X be a locally connected Hausdorff space and $W \subset X$ an open and connected subset. Suppose there exists open and connected subsets $U, V \subset W$ so that $W = U \cup V$ and $U \cap V$ is not connected. Then the first homology group $H_1(W; \mathbb{Z})$ is not trivial.*

Proof. Towards contradiction we suppose that $H_1(W; \mathbb{Z}) = 0$. Then the reduced Mayer-Vietoris sequence

$$\rightarrow H_1(W; \mathbb{Z}) \rightarrow \tilde{H}_0(U \cap V; \mathbb{Z}) \rightarrow \tilde{H}_0(U; \mathbb{Z}) \oplus \tilde{H}_0(V; \mathbb{Z}) \rightarrow \tilde{H}_0(W; \mathbb{Z})$$

takes the form

$$0 \rightarrow \tilde{H}_0(U \cap V; \mathbb{Z}) \rightarrow 0.$$

Thus, $\tilde{H}_0(U \cap V; \mathbb{Z}) = 0$. Thus $U \cap V$ is connected, which is a contradiction. Thus, $H_1(W; \mathbb{Z})$ is not trivial. \square

The following lemma is the key observation in the proof of Theorem 7.2.

Lemma 7.4. *Suppose $f: X \rightarrow Y$ is a branched covering between manifolds. Suppose W is a connected locally connected Hausdorff space and $p: W \rightarrow X$ and $q: W \rightarrow Y$ are normal branched coverings so that the diagram*

$$\begin{array}{ccc} & W & \\ p \swarrow & & \searrow q \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. Suppose there exists an open and connected subset $C_1 \subset Y$ so that $D_1 = f^{-1}(C_1)$ is connected and $f|_{D_1}: D_1 \rightarrow C_1$ is a normal branched covering. Then f is a normal branched covering, if $E_1 = q^{-1}(C_1)$ is connected.

Proof. Since $E_1 := q^{-1}(C_1)$ is connected, we have

$$\text{Deck}(q) = \{\tau \in \text{Deck}(q) : \tau(E_1) = E_1\} \cong \text{Deck}(q|_{E_1}: E_1 \rightarrow C_1)$$

and

$$\text{Deck}(p) = \{\tau \in \text{Deck}(p) : \tau(E_1) = E_1\} \cong \text{Deck}(p|_{E_1}: E_1 \rightarrow D_1),$$

where the isomorphisms are canonical in the sense that they map every deck-homomorphism $\tau: W \rightarrow W$ to the restriction $\tau|_{E_1}: E_1 \rightarrow E_1$.

Since $f|D_1: D_1 \rightarrow C_1$ is a normal branched covering,

$$\text{Deck}(p|E_1: E_1 \rightarrow D_1) \subset \text{Deck}(q|E_1: E_1 \rightarrow C_1)$$

is a normal subgroup. Hence, $\text{Deck}(p) \subset \text{Deck}(q)$ is a normal subgroup. Hence, the branched covering $f: X \rightarrow Y$ is normal. \square

Proof of Theorem 7.2. Let $W \subset X_f$ be a open set and $y \in f(W)$ a destructive point and $x \in \bar{f}^{-1}\{y\}$. By Lemma 7.3, to show that $H_1(W; \mathbb{Z}) \neq 0$ it is sufficient to show that there exists a domain cover of W that is not strong.

Let V_0 be a destructive neighbourhood of y so that the x -component W_0 of V_0 is a normal neighbourhood of x in W . Let $\{A, B\}$ be a strong domain cover of V_0 so that $y \in \bar{B} \subset V_0$ and $\{W_0^A, W_0^B\}$ is a domain cover of W_0 for $W_0^A := (f|W_0)^{-1}(A)$ and $W_0^B := (f|W_0)^{-1}(B)$, (see Lemma 2.2).

We first show that $\{W_0^A, W_0^B\}$ is not strong. Suppose towards contradiction that $\{W_0^A, W_0^B\}$ is strong. Then $A \cap B$, $f^{-1}(A) \cap f^{-1}(B)$ and $W_0^A \cap W_0^B$ are connected and

$$\begin{array}{ccc} & W_0^A \cap W_0^B & \\ p|W_0^A \cap W_0^B \swarrow & & \searrow \bar{f}|W_0^A \cap W_0^B \\ f^{-1}(A) \cap f^{-1}(B) & \xrightarrow{f|f^{-1}(A) \cap f^{-1}(B)} & A \cap B \end{array}$$

is a commutative diagram of branched covers. In particular, since $f|f^{-1}(A) \cap f^{-1}(B)$ is a normal branched cover $\text{Deck}(p|W_0^A \cap W_0^B) \subset \text{Deck}(\bar{f}|W_0^A \cap W_0^B)$ is a normal subgroup. On the other hand, since

$$W_0^A \cap W_0^B = (\bar{f}|W_0)^{-1}(A \cap B) = (p|W_0)^{-1}(f^{-1}(A) \cap f^{-1}(B)) \subset W_0$$

is connected, $\text{Deck}(p|W_0^A \cap W_0^B) \cong \text{Deck}(p|W_0)$, $\text{Deck}(\bar{f}|W_0^A \cap W_0^B) \cong \text{Deck}(\bar{f}|W_0)$ and in particular, $\text{Deck}(p|W_0) \subset \text{Deck}(\bar{f}|W_0)$ is a normal subgroup. Thus the factor $f|f^{-1}(V_0): f^{-1}(f(V_0)) \rightarrow V_0$ of $\bar{f}|W_0: W_0 \rightarrow V_0$ is a normal branched covering. This is a contradiction, since V_0 is a destructive neighbourhood of y and we conclude that $W_0^A \cap W_0^B \subset W_0$ is not connected.

Since $\bar{B} \subset V_0$ there exists a connected neighbourhood $W' \subset W$ of $W \setminus W_0^B$ so that $W' \cap W_0^B = \emptyset$. Now $W_0^A \cup W'$ is connected, since W_0^A is connected and every component of W' has a non-empty intersection with W_0^A . Further, $W = W_0^B \cup (W_0^A \cup W')$ and $W_0^B \cap (W_0^A \cup W') = W_0^A \cap W_0^B$. Thus $\{W_0^B, W_0^A \cup W'\}$ is a domain cover of W that is not strong and by Lemma 7.3 we conclude $H_1(W; \mathbb{Z}) \neq 0$. \square

This concludes the proof of Theorem 6.2, and further by Corollary 6.3, the proof of Theorem 1.2 in the introduction.

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